NORTH-HOLLAND

Spectral Inequalities for Generalized Rayleigh Quotients*

Rosa Amélia Martins
Departamento de Matemática
Universidade de Aveiro
3800 Aveiro, Portugal

and

João Filipe Queiró
Departamento de Matemática
Universidade de Coimbra
Apartado 3008
3000 Coimbra, Portugal

Submitted by Raphael Loewy

ABSTRACT

We show that the s-spaces of Carlson and Sá, with a few natural axioms added, are rich enough to develop unified abstract versions of theorems leading to very general families of inequalities involving eigenvalues of sums of Hermitian matrices, singular values of products of complex matrices, and invariant factors of products of matrices over principal-ideal domains.

1. INTRODUCTION

In the last 10–15 years, several authors, starting with R. C. Thompson [8], have remarked on the analogy that is exhibited, in various situations, by eigenvalues of Hermitian matrices, singular values of arbitrary complex

*Research supported by projects PBIC/C/CEN/1129/92 and STRDA/P/CEN/529/92.

655 Avenue of the Americas, New York, NY 10010

0024-3795/95/$9.50
SSDI 0024-3795(94)00110-Y

matrices, and invariant factors of matrices over principal-ideal domains. In a very interesting paper [1], D. Carlson and E. Marques de Sá invented an abstract structure, s-spaces, as a setting in which to prove unified versions of certain results concerning those quantities. They were especially concerned with interlacing theorems (relating matrices and submatrices), and their approach was based on Courant-Fischer-type formulas, proved for certain mappings ("generalized Rayleigh quotients") with values in a lattice. This is summarized below.

In this paper we propose to carry that kind of investigation further. The concrete problems that interest us are the description of the possible eigenvalues of a sum of two Hermitian matrices with given eigenvalues, of the possible singular values of a product of two complex matrices with given singular values, and of the possible invariant factors of a product of two matrices over a p.i.d. with given invariant factors. These are difficult (and fundamental) problems in matrix theory which, as far as we know, remain open. Each has been the object of considerable attention, from several points of view (some rather deep).

A landmark reference concerning the first of these three problems is a 1962 paper by A. Horn [2]. In Section 2 of that paper, an ingenious procedure was developed to obtain, from known inequalities, new, generally valid inequalities for eigenvalues of sums [2, Theorem 5]. This result is credited by Horn to a suggestion of Alan Hoffman. Prof. Hoffman has kindly informed us that the idea was in fact due to H. Wielandt. Analogous results were proved for singular values of products in [7, paper III, Theorem 4] and for invariant factors of products in [5]. These procedures can be used iteratively, in all three settings, to obtain large families of inequalities for the respective problems. This strategy led Horn to formulate a complicated conjecture, still unsettled, for the problem of eigenvalues of sums [2, p. 236].

Here we shall show that the s-spaces of Carlson and Sá, with a few natural axioms added, are rich enough to develop a unified abstract version of the above-mentioned procedures. It turns out that the emphasis on Rayleigh quotients (as opposed to the matrices themselves) leads to a simplification of the proofs in [7, paper III] and [5].

The main point of our paper is not so much building an abstract theory as showing that the results in [2; 7, paper III; 5], from an appropriate point of view, really are one and the same.

2. s-SPACES AND DIAGONALIZABLE MAPPINGS

We briefly review the main concepts introduced in [1].

Let V be a nonempty set and n a positive integer. For each i, $0 \leq i \leq n$, a family $\mathscr{S}_i$ of subsets of V is given (we think of the elements of $\mathscr{S}_i$ as "subspaces of dimension i"). For each $G \subseteq V$ we define its "dimension" as $d(G) = \min\{i : \text{there exists } F \in \mathscr{S}_i \text{ such that } F \supseteq G\}$.

We say V is an *s-space of dimension n* if:

(1) $\varnothing \in \mathscr{S}_0$, $V \in \mathscr{S}_n$.
(2) $F \in \mathscr{S}_i \Rightarrow d(F) = i$.
(3) If F and G are subspaces, $d(F \cap G) + d(F \cup G) \geq d(F) + d(G)$.

Next, let L be a partially ordered set (with reflexive order $\prec$) which is a conditionally complete lattice, that is, a lattice such that bounded subsets have inf and sup.

If V is an s-space of dimension n, a mapping ϕ: $V \to L$ is said to be *diagonalizable* if:

(1) $\phi(V)$ is bounded.
(2) There exist chains $X_1 \subset \cdots \subset X_n$, $X_i \in \mathscr{S}_i$, and $X_1' \supset \cdots \supset X_n'$, $X_i' \in \mathscr{S}_{n-i+1}$, such that

$$\inf_{x \in X_i} \phi(x) = \sup_{x \in X'_{n-i+1}} \phi(x), \qquad 1 \leq i \leq n.$$

For convenience we call these chains of subspaces *proper chains* of ϕ.

3. MINIMAX AND INTERLACING

MINIMAX THEOREM [1, Theorem 2.1]. *In the above situation we have*

$$\sup_{X \in \mathscr{S}_i} \inf_{x \in X} \phi(x) = \inf_{X' \in \mathscr{S}_{n-i+1}} \sup_{x \in X'} \phi(x), \qquad 1 \leq i \leq n.$$

If we denote by α_i the element of L defined by the above equality, $1 \leq i \leq n$, it is clear that $\alpha_1 \succ \cdots \succ \alpha_n$. These are called the *invariants* of ϕ.

Notice now that each $W \in \mathscr{S}_m$, $m < n$, is an s-space, taking the family of subspaces of V which are contained in W.

INTERLACING THEOREM [1, Theorem 2.2]. *If ϕ: $V \to L$ is diagonalizable with invariants $\alpha_1 \succ \cdots \succ \alpha_n$, W is a subspace of V of dimension $n - p$, and $\phi|_W$ is diagonalizable with invariants $\alpha_1' \succ \cdots \succ \alpha_{n-p}'$, then $\alpha_i \succ \alpha_i' \succ \alpha_{i+p}$, $1 \leq i \leq n - p$.*

4. ADDITIONAL STRUCTURE AND RESULTS

We begin with a technical lemma.

LEMMA 1. *Let V be an s-space of dimension $n + 1$. Let ϕ: $V \to L$ be diagonalizable, with invariants $\alpha_1 \succ \cdots \succ \alpha_{n+1}$ and proper chains $X_1 \subset \cdots \subset X_{n+1}$ and $X'_1 \supset \cdots \supset X'_{n+1}$. Let W be a subspace of V of dimension n, and suppose $\phi|_W$ is diagonalizable with invariants $\alpha'_1 \succ \cdots \succ \alpha'_n$. Given $k \in \{1, \ldots, n\}$,*

(1) *if $X_k \subseteq W$ then $\alpha'_i = \alpha_i$, $i = 1, \ldots, k$;*
(2) *if $X'_{k+1} \subseteq W$ then $\alpha'_i = \alpha_{i+1}$, $i = k, \ldots, n$.*

Proof. (1): By the interlacing theorem, $\alpha'_i \prec \alpha_i$ for all admissible i. Now, for $1 \le i \le k$,

$$\alpha'_i = \sup_{X \in \mathscr{S}_i,\ X \subseteq W} \inf_{x \in X} \phi(x) \succ \inf_{x \in X_i} \phi(x) = \alpha_i,$$

since $X_i \in \mathscr{S}_i$, $X_i \subseteq W$.

(2): By the interlacing theorem, $\alpha'_i \succ \alpha_{i+1}$ for all admissible i. Now, for $k \le i \le n$,

$$\alpha'_i = \inf_{X' \in \mathscr{S}_{n-i+1},\ X' \subseteq W} \sup_{x \in X'} \phi(x) \prec \sup_{x \in X'_{i+1}} \phi(x) = \alpha_{i+1},$$

since $X'_{i+1} \in \mathscr{S}_{n-i+1}$, $X'_{i+1} \subseteq W$. ■

From now on we suppose given in L an associative binary operation $(\alpha, \beta) \to \alpha \cdot \beta$, compatible with the order $\prec$, and with left and right identity element θ. We assume also that there exists in L an element ξ such that the elements in the sequence θ, ξ, $\xi \cdot \xi$, $\xi \cdot \xi \cdot \xi, \ldots$ are decreasing (with respect to $\prec$) and distinct.

For each s-space V considered below, we assume the following:

(1) For F, G subspaces of V, $d(F \cup G) \le d(F) + d(G)$.

(2) For $F \subseteq V$, if $d(F) \le k$, there exists $W \in \mathscr{S}_k$ such that $F \subseteq W$.

(3) A family Φ of mappings from V to L is given such that each element of Φ is diagonalizable on every subspace W of V and, additionally, if $\phi \in \Phi$ and W and W' are subspaces of V with the same dimension, then $\phi|_W$ and $\phi|_{W'}$ have the same invariants.

(4) Given elements $\mu_1 \succ \cdots \succ \mu_m$ of L and a subspace W of V with dimension m, there exists $\phi \in \Phi$ such that $\phi|_W$ has invariants $\mu_1 \succ \cdots \succ \mu_m$.

(5) A family of mappings indexed by Φ, $\Lambda = \{\lambda_\psi: \psi \in \Phi\}$, is given, where each element of Λ is a mapping $\lambda_\psi: V \to V$ such that, for any $\phi \in \Phi$, $\phi \circ \lambda_\psi$ is still in Φ and, for any subspace X of V, $\lambda_\psi^{-1}(X)$ is a subspace with the same dimension as X and $\phi(X \cap \operatorname{Im} \lambda_\psi) = \phi(X)$.

(6) The constant mapping $\Theta: V \to L$ given by $\Theta(x) = \theta$, all $x \in V$, belongs to Φ (clearly, all its invariants are equal to θ), and λ_Θ is the identity mapping on V.

(7) In Φ we have a binary operation $(\phi, \psi) \to \phi * \psi$ for which Θ is a left and right identity element.

(Condition (5) is adapted from [1, Section 4].)

We are interested in certain inequalities relating the invariants of ϕ, ψ, and $\phi * \psi$.

For $r \in \{1, \ldots, m\}$, consider sequences $\mathbf{i} = (i_1, \ldots, i_r)$, $\mathbf{j} = (j_1, \ldots, j_r)$, and $\mathbf{k} = (k_1, \ldots, k_r)$ with $1 \le i_1 < \cdots < i_r \le m$, $1 \le j_1 < \cdots < j_r \le m$, and $1 \le k_1 < \cdots < k_r \le m$.

DEFINITION. Let V be an s-space. Given $m \le d(V)$, Σ_r^m is the set of triples of sequences $(\mathbf{i}; \mathbf{j}; \mathbf{k})$ such that the inequality

$$\gamma_{k_1} \cdot \cdots \cdot \gamma_{k_r} \prec \alpha_{i_1} \cdot \cdots \cdot \alpha_{i_r} \cdot \beta_{j_1} \cdot \cdots \cdot \beta_{j_r}$$

holds whenever $\alpha_1 \succ \cdots \succ \alpha_m$, $\beta_1 \succ \cdots \succ \beta_m$, and $\gamma_1 \succ \cdots \succ \gamma_m$ are the invariants of $\phi \circ \lambda_\psi|_W$, $\psi|_W$, and $\phi * \psi|_W$, respectively, ϕ and ψ arbitrary in Φ, and W a subspace of V with dimension m.

(Notice that $\phi \circ \lambda_\psi$ has the same invariants as ϕ.)

LEMMA 2. *Let V be an s-space, $m \le d(V)$. If $(\mathbf{i}; \mathbf{j}; \mathbf{k}) \in \Sigma_r^m$, then for every $p \in \{1, \ldots, r\}$ we have $i_p \le k_p$ and $j_p \le k_p$.*

Proof. We prove that $i_p \le k_p$ (the reasoning for the other inequality is similar). Let W be a subspace of V with dimension m. Take the special element $\xi \in L$ mentioned above. Let $\phi \in \Phi$ be such that $\phi|_W$ has invariants $\alpha_1 = \cdots = \alpha_{k_p} = \theta$ and $\alpha_{k_p+1} = \cdots = \alpha_m = \xi$. Take $\psi = \Theta$. Then $\phi * \psi = \phi$ and so we have

$$\alpha_{k_1} \cdot \cdots \cdot \alpha_{k_r} \prec \alpha_{i_1} \cdot \cdots \cdot \alpha_{i_r}.$$

The left-hand side is equal to $\xi \cdot \cdots \cdot \xi$ ($r - p$ times). If $i_p > k_p$, the right-hand side is $\prec \xi \cdot \cdots \cdot \xi$ ($r - p + 1$ times), and the inequality contradicts the property assumed for ξ. ∎

THEOREM 1. *Let V be an s-space of dimension $n+1$. If $(\mathbf{i};\mathbf{j};\mathbf{k}) \in \Sigma_r^n$, then $(\mathbf{i};\mathbf{j};\mathbf{k}) \in \Sigma_r^{n+1}$.*

Proof. Let $\alpha_1 \succ \cdots \succ \alpha_{n+1}$ and $\beta_1 \succ \cdots \succ \beta_{n+1}$ be the invariants of arbitrary ϕ, $\psi \in \Phi$, respectively. Let $\gamma_1 \succ \cdots \succ \gamma_{n+1}$ be the invariants of $\phi * \psi$. Denote by $W_1 \subset \cdots \subset W_{n+1}$ an ascending proper chain of $\phi * \psi$. Let $\alpha_1' \succ \cdots \succ \alpha_n'$, $\beta_1' \succ \cdots \succ \beta_n'$, and $\gamma_1' \succ \cdots \succ \gamma_n'$ be the invariants of $\phi \circ \lambda_\psi |_{W_n}$, $\psi|_{W_n}$, and $\phi * \psi|_{W_n}$, respectively. For $i = 1, \dots, n$ we have $\alpha_i' \prec \alpha_i$ and $\beta_i' \prec \beta_i$ by interlacing, and $\gamma_i' = \gamma_i$ by Lemma 1. It follows, since $(\mathbf{i};\mathbf{j};\mathbf{k}) \in \Sigma_r^n$, that

$$\begin{aligned}\gamma_{k_1} \cdot \cdots \cdot \gamma_{k_r} = \gamma_{k_1}' \cdot \cdots \cdot \gamma_{k_r}' &\prec \alpha_{i_1}' \cdot \cdots \cdot \alpha_{i_r}' \cdot \beta_{j_1}' \cdot \cdots \cdot \beta_{j_r}' \\ &\prec \alpha_{i_1} \cdot \cdots \cdot \alpha_{i_r} \cdot \beta_{j_1} \cdot \cdots \cdot \beta_{j_r}. \qquad \blacksquare\end{aligned}$$

We now come to our main result.

THEOREM 2. *Let V be an s-space of dimension $n+1$. Let $k_r = n$. Suppose that $(\mathbf{i};\mathbf{j};\mathbf{k}) \in \Sigma_r^n$. Take $u, v \in \{1, \dots, n+1\}$, $w \in \{1, \dots, n\}$. If $i_u + j_v \geq k_{w-1} + k_r + 2$, then the sequence*

$$\begin{aligned}(i_1, \dots, i_{u-1}, i_u + 1, \dots, i_r + 1; j_1, \dots, j_{v-1}, j_v + 1, \dots, j_r + 1;\\ k_1, \dots, k_{w-1}, k_w + 1, \dots, k_r + 1)\end{aligned}$$

belongs to Σ_r^{n+1}. (Here, by convention, $k_0 = 0$ and $i_{r+1} = j_{r+1} = n+1$.)

Proof. Let $\alpha_1 \succ \cdots \succ \alpha_{n+1}$ and $\beta_1 \succ \cdots \succ \beta_{n+1}$ be the invariants of arbitrary ϕ, $\psi \in \Phi$, respectively. Let $\gamma_1 \succ \cdots \succ \gamma_{n+1}$ be the invariants of $\phi * \psi$. Denote by $X_1' \supset \cdots \supset X_{n+1}'$ and $Y_1' \supset \cdots \supset Y_{n+1}'$ descending proper chains if $\phi \circ \lambda_\psi$ and ψ, respectively, and by $W_1 \subset \cdots \subset W_{n+1}$ an ascending proper chain of $\phi * \psi$. We have

$$d\left(X_{i_u+1}' \cup Y_{j_v+1}' \cup W_{k_{w-1}}\right) \leq d\left(X_{i_u+1}'\right) + d\left(Y_{j_v+1}'\right) + d\left(W_{k_{w-1}}\right) \leq n$$

by the hypothesis. Let W be a subspace of V of dimension n containing $X_{i_u+1}' \cup Y_{j_v+1}' \cup W_{k_{w-1}}$. Let $\alpha_1' \succ \cdots \succ \alpha_n'$, $\beta_1' \succ \cdots \succ \beta_n'$, and $\gamma_1' \succ \cdots \succ \gamma_n'$ be the invariants of $\phi \circ \lambda_\psi|_W$, $\psi|_W$, and $\phi * \psi|_W$, respectively. By interlacing we have $\alpha_i' \prec \alpha_i$, $\beta_j' \prec \beta_j$, and $\gamma_{k+1} \prec \gamma_k'$ for all admissible i, j, and k. By Lemma 1 we have $\alpha_i' = \alpha_{i+1}$ for $i = i_u, \dots, n$, $\beta_j' = \beta_{j+1}$ for

$j = j_v, \ldots, n$, and $\gamma'_k = \gamma_k$ for $k = 1, \ldots, k_{w-1}$. It follows, since $(\mathbf{i}; \mathbf{j}; \mathbf{k}) \in \Sigma_r^n$, that

$$\begin{aligned}
&\gamma_{k_1} \cdot \cdots \cdot \gamma_{k_{w-1}} \cdot \gamma_{k_w+1} \cdot \cdots \cdot \gamma_{k_r+1} \\
&\quad \prec \gamma'_{k_1} \cdot \cdots \cdot \gamma'_{k_r} \\
&\quad \prec \alpha'_{i_1} \cdot \cdots \cdot \alpha'_{i_r} \cdot \beta'_{j_1} \cdot \cdots \cdot \beta'_{j_r} \\
&\quad \prec \alpha_{i_1} \cdot \cdots \cdot \alpha_{i_{u-1}} \cdot \alpha_{i_u+1} \cdot \cdots \cdot \alpha_{i_r+1} \cdot \beta_{j_1} \cdot \cdots \cdot \beta_{j_{v-1}} \cdot \beta_{j_v+1} \cdot \cdots \cdot \beta_{j_r+1}. \quad \blacksquare
\end{aligned}$$

5. SOME COMBINATORICS WITH SEQUENCES

Dropping now any reference to s-spaces, diagonalizable mappings, and invariants, we show how sets of triples of sequences like those just considered can be related in a purely combinatorial way. Specifically, we show that the existence of special triples in one set implies that certain families of triples belong to other sets. This can be applied to concrete settings in the production of new inequalities from known ones (see the next section).

Fix $r \in \mathbf{N}$. For each $m \geq r$ consider sets Σ_r^m of sequences $(\mathbf{i}; \mathbf{j}; \mathbf{k})$ as in the previous section. Suppose that the following conditions are satisfied by these sets:

Condition 1. $\Sigma_r^n \subseteq \Sigma_r^{n+1}$ for all $n \geq r$.

Condition 2. If $(\mathbf{i}; \mathbf{j}; \mathbf{k}) \in \Sigma_r^{k_r}$ and $i_u + j_v \geq k_{w-1} + k_r + 2$ (where $u, v \in \{1, \ldots, k_r + 1\}$, $w \in \{1, \ldots, k_r\}$), then the sequence

$$(i_1, \ldots, i_{u-1}, i_u + 1, \ldots, i_r + 1; j_1, \ldots, j_{v-1}, j_v + 1, \ldots, j_r + 1; \\ k_1, \ldots, k_{w-1}, k_w + 1, \ldots, k_r + 1)$$

belongs to $\Sigma_r^{k_r+1}$. (Here, by convention, $k_0 = 0$ and $i_{r+1} = j_{r+1} = k_r + 1$.)

Then we have:

Theorem 3. *Let $n \geq r$. If $(1, \ldots, r; 1, \ldots, r; 1, \ldots, r) \in \Sigma_r^r$ and $1 \leq i_1 < \cdots < i_r \leq n$, $1 \leq j_1 < \cdots < j_r \leq n$ are any sequences with $i_r + j_r \leq n + r$, then the triple of sequences*

$$(i_1, \ldots, i_r; j_1, \ldots, j_r; i_1 + j_1 - 1, \ldots, i_r + j_r - r)$$

belongs to Σ_r^n.

Proof. Start with the sequence $(1,\dots,r;1,\dots,r;1,\dots,r)$. For $t = 0,1,\dots,r-2$ apply Condition 2 first $i_{r-t} - i_{r-t-1} - 1$ times with $u = w = r-t$, $v = r+1$, and then $j_{r-t} - j_{r-t-1} - 1$ times with $v = w = r-t$, $u = r+1$. This leads to the sequence

$$\begin{aligned}&(1, 1 + i_2 - i_1, \dots, 1 + i_r - i_1; 1, 1 + j_2 - j_1, \dots, 1 + j_r - j_1;\\ &1, i_2 - i_1 + j_2 - j_1, i_3 - i_1 + j_3 - j_1 - 1, \dots, i_r - i_1\\ &\qquad\qquad + j_r - j_1 - (r-2)),\end{aligned}$$

which belongs to $\Sigma_r^{i_r - i_1 + j_r - j_1 - (r-2)}$. Here the spacing between the indices is already as we want it. It just remains to displace them to the right. Apply Condition 2 $i_1 - 1$ times with $u = w = 1$, $v = r+1$, and then $j_1 - 1$ times with $v = w = 1$, $u = r+1$. We obtain

$$(i_1,\dots,i_r; j_1,\dots,j_r; i_1 + j_1 - 1, \dots, i_r + j_r - r) \in \Sigma_r^{i_r + j_r - r},$$

and the conclusion of the theorem follows from Condition 1. ∎

COROLLARY. *Let $n \geq r$. If $(1,\dots,r;1,\dots,r;1,\dots,r) \in \Sigma_r^r$ and $1 \leq i_1 < \cdots < i_r \leq n$ is any sequence, then the triples of sequences*

$$(i_1,\dots,i_r;1,\dots,r;i_1,\dots,i_r) \quad \textit{and} \quad (1,\dots,r;i_1,\dots,i_r;i_1,\dots,i_r)$$

belong to Σ_r^n.

6. APPLICATIONS

In this section we sketch the application of the previous results to the three problems mentioned at the beginning. We refer the reader to [1] for more details about the three concrete models.

Eigenvalues of Sums of Hermitian Matrices

Here the s-spaces are of the type $\mathbf{C}^n \setminus \{0\}$ with their subspaces. L is $\mathbf{R}$ with the usual order and the operation of addition. The elements of Φ are the Rayleigh quotients x^*Ax/x^*x, where A runs over the $n \times n$ Hermitian matrices, and $*$ is pointwise addition. Λ consists of just the identity mapping.

In this setting, the results in Section 4 are due to Horn [2]. The inequalities associated to the sequences in Theorem 3 were discovered by Thompson and Freede [6]. Their proof in this way was suggested by Oliveira

[4]. The less general inequalities associated to the sequences in the Corollary to Theorem 3 are the famous Lidskii-Wielandt inequalities [3, 11]. They were proved using this kind of argument already by Horn [2, Theorem 6].

Singular Values of Products of Complex Matrices

When studying inequalities valid for singular values of products of arbitrary complex matrices, it is clear we can restrict ourselves to square and (by continuity) nonsingular matrices. Again our abstract analysis is applied to the s-spaces $\mathbf{C}^n \setminus \{0\}$ with their subspaces. L is $\mathbf{R}^+$ with the usual order and the operation of multiplication. The elements of Φ are the quotients $\|Ax\|/\|x\|$, where A runs over the $n \times n$ upper triangular matrices with positive diagonal elements (i.e. the canonical forms for left unitary equivalence). In this way we have a bijection between matrices and their associated quotients. The operation $*$ is pointwise multiplication. As to the family Λ: if ψ is the quotient associated with B, then $\lambda_\psi(x)$ is by definition Bx.

In this setting, the results in Section 4 are due to Thompson and Therianos [7, paper III], with different (although related) proofs. The inequalities associated to the sequences in Theorem 3 appear in [7, paper I, II], with different proofs.

Invariant Factors of Products of Matrices over a p.i.d. R

When studying inequalities valid for invariant factors of products of arbitrary matrices over R, we can restrict ourselves to square and (by a technical argument in [5]) nonsingular matrices. Our abstract analysis is applied to the s-spaces $R^n \setminus \{0\}$ with their *pure* submodules (see [1, pp. 79–80]). L is a set of distinct representatives of the equivalence classes of associated elements of R, closed under multiplication; it is a complete lattice with the order being defined by $a \prec b \Leftrightarrow b \mid a$ [1, p. 78]. The elements of Φ are the quotients $\nu(Ax)/\nu(x)$ (see [1, p. 90] for the definition of ν), where A runs over the $n \times n$ nonsingular Hermite normal forms (canonical forms for left unimodular equivalence). In this way we have a bijection between matrices and their associated quotients (see [1, p. 91]). The operation $*$ is pointwise multiplication. As to the family Λ: if ψ is the quotient associated with B, then $\lambda_\psi(x)$ is by definition Bx.

Only the condition $\phi(X \cap \operatorname{Im} \lambda) = \phi(X)$ requires some attention here (in the other two situations it is trivially satisfied, as $\operatorname{Im} \lambda$ is the whole space). Let X be an arbitrary pure submodule. We notice that X coincides with the pure closure of $\lambda(\lambda^{-1}(X)) = X \cap \operatorname{Im} \lambda$: clearly the latter is contained in the former, and since λ is one-to-one they have the same dimension, whence the result follows by purity. We now prove the desired equality. Take $v \in X$. Then v belongs to the pure closure of $\lambda(\lambda^{-1}(X))$, so there exists $\alpha \in R$ such that $\alpha v \in \lambda(\lambda^{-1}(X))$, that is, $\alpha v = \lambda(u)$ with $\lambda(u) \in X$. Since clearly $\phi(v) = \phi(\alpha v)$, we have shown that $\phi(X) \subseteq \phi(X \cap \operatorname{Im} \lambda)$.

In this setting, the results in Section 4 are due to Sá [5], with different (although related) proofs. The inequalities associated to the sequences in Theorem 3 appear in [5] (for the proof, attention is called there to the suggestion in [4]). They were later published by Thompson [9, 10] with different proofs.

REFERENCES

1 D. Carlson and E. Marques de Sá, Generalized minimax and interlacing theorems, *Linear and Multilinear Algebra* 15:77–103 (1984).
2 A. Horn, Eigenvalues of sums of Hermitian matrices, *Pacific J. Math.* 12:225–241 (1962).
3 V. B. Lidskii, On the proper values of the sum and product of symmetric matrices (in Russian), *Dokl. Akad. Nauk SSSR* 75:769–772 (1950).
4 G. N. de Oliveira, Review of [6], *Zbl.* 228:79 (1972).
5 E. Marques de Sá, Interlacing problems for invariant factors (in Portuguese), Coimbra, 1978 (unpublished).
6 R. C. Thompson and L. Freede, On the eigenvalues of sums of Hermitian matrices, *Linear Algebra Appl.* 4:369–376 (1971).
7 R. C. Thompson and S. Therianos. On the singular values of matrix products—I, II, III, *Scripta Math.* 29:99–123 (1973).
8 R. C. Thompson, Interlacing inequalities for invariant factors, *Linear Algebra Appl.* 24:1–32 (1979).
9 R. C. Thompson, An inequality for invariant factors, *Proc. Amer. Math. Soc.* 86:9–11 (1982).
10 R. C. Thompson, Smith invariants of a product of integral matrices, *Contemp. Math.* 47:401–435 (1985).
11 H. Wielandt, An extremum property of sums of eigenvalues, *Proc. Amer. Math. Soc.* 6:106–110 (1955).

Received 9 November 1993; final manuscript accepted 8 May 1994